NASA TT F-9334

NASA TT F-9334

THE RAYLEIGH-RITZ METHOD WITH HERMITIAN INTERPOLATION POLYNOMIALS

S. Falk

FACILITY FORM 602

N65-23676 (ACCESSION NUMBER)

(THRU)

33 (PAGES)

1 (CODE)

(NASA CR OR TMX OR AD NUMBER)

19 (CATEGORY)

Translation of "Das Verfahren von Rayleigh-Ritz mit hermiteschen Interpolationspolynomen". Zeitschrift für Angewandte Mathematik und Mechanik, Vol. 43, No. 4/5, pp. 149-166, 1963.

GPO PRICE $ ______

OTS PRICE(S) $ ______

Hard copy (HC) $2.00

Microfiche (MF) .50

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
WASHINGTON MAY 1965

NASA TT F-9334

THE RAYLEIGH-RITZ METHOD WITH HERMITIAN INTERPOLATION POLYNOMIALS *[/149]

S. Falk

23676

Hermitian interpolation polynomials are applied to the development of admissible or comparison functions for the Ritz method, for simplification of the calculation of elastically supported beams, harmonically vibrating at constant pressure (two-parameter eigenvalue problem). For differential equations with constant coefficients, ready-made integration matrices are given such that the entire method is reduced to a mere multiplication of matrices and, consequently, readily programmed on digital computers.

AUTHOR

1. Mathematical Principles

Given is the linear ordinary differential equation of the order 2n in the so-called self-conjugate form

$$L[y] \equiv (-1)^n (g_n y^{(n)})^{(n)} + \cdots + (g_2 y'')'' - (g_1 y')' + g_0 y = r \qquad (1.1)$$

with logical functions $g_i(x)$ and $r(x)$** and 2n linear boundary conditions for $x = 0$ and $x = \ell$, in the form of

(1.2)

where $\mathfrak{h}_0$ and $\mathfrak{h}_\ell$ denote the vectors of the boundary derivatives***

---

* Numbers in the margin indicate pagination in the original foreign text.

** Details for this case and for all problems in this Section are given by Collatz (Bibl.1).

*** The symbol * indicates transposition of a matrix or of a vector.

$$\begin{cases} \mathfrak{y}_0^* = (y_0,\ y_0',\ y_0'',\ \ldots,\ y_0^{(2n-1)}), \\ \mathfrak{y}_l^* = (y_l,\ y_l',\ y_l'',\ \ldots,\ y_0^{(2n-1)}), \end{cases} \tag{1.3}$$

and where $\mathfrak{R}_0$ and $\mathfrak{R}_\ell$ are two square matrices of the order 2n. Of the rectangular total matrix

(1.4)

it is required only that column regularity exists, i.e., that linear independence of the boundary conditions (1.2) is present. If the two boundaries are not coupled, eq.(1.2) will have the special form

(1.5)

which, however, is of no importance for what follows.

Now, let $\Pi = \Pi_d + \Pi_k$ be a certain "energy expression" (in mechanics, for example, $\Pi = W - A$ where W is the work of deformation and A the work done by the load) where Π_d is due to the discrete boundary values and Π_k to the continuous deposition in the field $0 \le x \le \ell$. Both components are assumed to be at most quadratic in the functions $y, y', y'', \ldots, y^{(n)}$, respectively in their boundary derivatives combined into the vector

(1.6)

at the points $x = 0$ and $x = \ell$: /150

$$\Pi_k = \frac{1}{2}\int_0^l (g_n\, y^{(n)2} + \cdots + g_2\, y''^2 + g_1\, y'^2 + g_0\, y^2)\, dx - \int_0^l r\, y\, dx + \int_0^l f\, dx \tag{1.7}$$

$$\Pi_d = \frac{1}{2}\mathfrak{z}^*\,\mathfrak{A}\,\mathfrak{z} - \mathfrak{z}^*\,\mathfrak{f} + f_d \quad \text{at} \quad \mathfrak{A}^* = \mathfrak{A}. \tag{1.8}$$

If, then, the expression $\Pi = \Pi_d + \Pi_k$ is made into the extremum with reference to all admissible functions $y(x)$ - these are functions that satisfy all important (geometric) boundary conditions - , then the expressions

$$L[y] - r = 0 \tag{1.9}$$

satisfaction of the residual (dynamic) boundary conditions (1.10) are necessary conditions for the wanted extremals; this fact was utilized by Ritz in the following manner: First, the energy term Π, by means of a linear argument

$$y(x) = a_1 v_1(x) + \cdots + a_\varrho v_\varrho(x) + d v_{\varrho+1}(x) \tag{1.11}$$

of admissible functions $v_i(x)$ with still free constants a_i but fixed d is transformed into an equivalent energy $\widetilde{\Pi}$ consisting of quadratic forms and scalar products. If eq.(1.11) is abbreviated to

$$y(x) = \mathfrak{a}^* \mathfrak{v}(x) \tag{1.12}$$

$$\left\{ \begin{array}{l} \mathfrak{R}_0 \mathfrak{y}_0 = \mathfrak{d}_0, \\ \mathfrak{R}_1 \mathfrak{y}_1 = \mathfrak{d}_1, \end{array} \right.$$

with

$$\mathfrak{a}^* = (a_1, a_2, \ldots, a_\varrho; d)$$

and

$$\mathfrak{v}^*(x) = (v_1(x), v_2(x), \ldots, v_\varrho(x); v_{\varrho+1}(x)), \tag{1.14}$$

then a substitution of eq.(1.2) into eq.(1.7) will yield directly

$$\widetilde{\Pi}_k = \frac{1}{2} \sum_{\nu=0}^{n} \int_0^l g_\nu \mathfrak{a}^* \mathfrak{v}^{(\nu)} \mathfrak{v}^{(\nu)*} \mathfrak{a} \cdot dx - \int_0^l r \mathfrak{a}^* \mathfrak{v}\, dx + \int_0^l f\, dx = \frac{1}{2} \mathfrak{a}^* \sum_{\nu=0}^{n} \mathfrak{G}_\nu \cdot \mathfrak{a} - \mathfrak{a}^* \mathfrak{r} + f_k \tag{1.15}$$

with

$$\mathfrak{G}_\nu = \int_0^l g_\nu(x)\, \mathfrak{v}^{(\nu)}(x)\, \mathfrak{v}^{(\nu)*}(x) \cdot dx; \quad \mathfrak{G} = \sum_{\nu=0}^{n} \mathfrak{G}_\nu \tag{1.16}$$

$$\mathfrak{r} = \int_0^l r(x)\, \mathfrak{v}(x)\, dx \tag{1.17}$$

$$f_k = \int_0^l f(x)\, dx = \text{const}. \tag{1.18}$$

The energy Π_d [eq.(1.8)] has been transformed by the argument (1.12) into

$$\widetilde{\Pi}_d = \frac{1}{2} \mathfrak{a}^* \mathfrak{F} \mathfrak{a} - \mathfrak{a}^* \mathfrak{z} + f_d \tag{1.19}$$

which means that the total equivalent energy will then read

$$\widetilde{\Pi} = \widetilde{\Pi}_d + \widetilde{\Pi}_k = \frac{1}{2} \mathfrak{a}^* (\mathfrak{F} + \mathfrak{G})\, \mathfrak{a} - \mathfrak{a}^* (\mathfrak{r} + \mathfrak{z}) + (f_d + f_k), \tag{1.20}$$

This equivalent energy was used by Ritz for the extremum with reference to the still variable coefficients a_i:

$$\operatorname{grad} \widetilde{\Pi} = \operatorname{grad} \widetilde{\Pi}_d + \operatorname{grad} \widetilde{\Pi}_k = (\hat{\mathfrak{F}}\, \mathfrak{a} - \hat{\mathfrak{s}}) + (\hat{\mathfrak{G}}\, \mathfrak{a} - \hat{\mathfrak{r}}) = 0 \tag{1.21}$$

or

$$\mathfrak{L}\, \mathfrak{a} \equiv (\hat{\mathfrak{F}} + \hat{\mathfrak{G}})\, \mathfrak{a} = \hat{\mathfrak{s}} + \hat{\mathfrak{r}}\,, \tag{1.22}$$

where the symbol ^ means that the last rows of the matrices $\mathfrak{F}$ and $\mathfrak{G}$ as well as the last element of the vectors $\mathfrak{s}$ and $\mathfrak{r}$ must be deleted since d had been a fixed constant. Thus, the linear system of equations (1.22) represents a finite transposition of the differential equation (1.1). If the coefficients a_i are calculated from eq.(1.22), then the extremal of the equivalent problem will be available, according to eq.(1.11), as a function of x.

Of special interest is also the so-called holohomogeneous case: since $r(x) \equiv 0$ in eq.(1.1) and $\mathfrak{d} = 0$ in eq.(1.2), it follows that $d = 0$ in the argument (1.11) and $\mathfrak{r} = 0$ in eq.(1.17). Similarly, $\mathfrak{t}$ in eq.(1.8) will also vanish. So as to have any nontrivial solutions exist at all, at least one of the differential expressions on the left-hand side of eq.(1.1) must contain a factor λ, so that eqs.(1.1) and (1.2) can be written in the form of /151

$$L[y] = M[y] - \lambda\, N[y] = 0\,, \tag{1.23}$$

$$\mathfrak{R}_0\, \mathfrak{y}_0 + \mathfrak{R}_l\, \mathfrak{y}_l = 0 \tag{1.24}$$

In that case, generally ∞^1 discrete eigenvalues λ_i with the corresponding eigenfunctions $y_i(x)$ will exist. The eigenvalues are real and positive if the differential expressions

$$M[y] \quad \text{and} \quad N[y] \tag{1.25}$$

are symmetric and positive definite. The finite transformation of eqs.(1.23) and (1.24) will then read

$$\mathfrak{L}\, \mathfrak{a} \equiv \mathfrak{M}\, \mathfrak{a} - \Lambda\, \mathfrak{N}\, \mathfrak{a} = 0\,, \tag{1.26}$$

and the corresponding characteristic equation

$$|\mathfrak{M} - \Lambda\, \mathfrak{N}| = p(\Lambda) = s_\varrho\, \Lambda^\varrho + \cdots + s_2\, \Lambda^2 + s_1\, \Lambda + s_0 = 0 \tag{1.27}$$

will then yield ρ approximation eigenvalues Λ_i which also all are real and positive since the linear argument (1.2) transfers the properties (1.25) also to the matrix pair $\mathfrak{M}$; $\mathfrak{N}$. If the wanted eigenvalues λ_i and the approximation values Λ_i are arranged in order of magnitude, the following will apply under the assumptions of eq.(1.25):

$$\lambda_i \leqq \Lambda_i \text{ for } i = 1, 2, 3, \ldots \quad - \rho - \tag{1.28}$$

A single-term argument $y(x) = a_1 v_1(x)$ will change eq.(1.26) into the expression

$$(m_{11} - \Lambda\, n_{11})\, a_1 = 0 \quad \text{i.e.,}\ \Lambda = \frac{m_{11}}{n_{11}} \geqq \lambda_1,\ a_1 \ \textbf{arbitrary.} \tag{1.29}$$

In this form, Λ is known as the so-called Rayleigh quotient (called also "energy quotient" in the technical literature). In conclusion, we assemble the entire method schematically:

Given Problem		Equivalent Problem
$\Pi_k = \frac{1}{2}\sum_0^n \int_0^l g_\nu y^{(\nu)2}\, dx - \int_0^l r\, y\, dx + \int_0^l f\, dx$ $\Pi_d = \frac{1}{2}\mathfrak{z}^* \mathfrak{A}\, \mathfrak{z} - \mathfrak{z}^* \mathfrak{k} + f_d$	Argument $y(x) = \mathfrak{a}^* \mathfrak{v}(x)$	$\tilde{\Pi}_k = \frac{1}{2}\mathfrak{a}^* \sum_0^n \mathfrak{G}_\nu\, \mathfrak{a} - \mathfrak{a}^* \mathfrak{r} + \int_0^l f\, dx$ $\tilde{\Pi}_d = \frac{1}{2}\mathfrak{a}^* \mathfrak{F}\, \mathfrak{a} - \mathfrak{a}^* \mathfrak{s} + f_d$
	The requirement	
$\Pi = \Pi_k + \Pi_d = \text{Extremum}$		$\tilde{\Pi} = \tilde{\Pi}_k + \tilde{\Pi}_d = \text{Extremum}$
	with reference to all	
permissible functions $y(x)$		variables a_i
	leads to the necessary condition	
$\delta\Pi = \delta\Pi_k + \delta\Pi_d = 0$		$\operatorname{grad}\tilde{\Pi} = \operatorname{grad}\tilde{\Pi}_k + \operatorname{grad}\tilde{\Pi}_d = 0$
	i.e., to	
the linear differential equation $L[y] \equiv \cdots + (g_2 y'')'' - (g_1 y')' + g_0 y = r$ (and residual boundary conditions).		the linear system of equations $L\,\mathfrak{a} \equiv (\cdots + \hat{\mathfrak{G}}_2 + \hat{\mathfrak{G}}_1 + \hat{\mathfrak{G}}_0 + \hat{\mathfrak{F}})\, \mathfrak{a} = \hat{\mathfrak{r}} + \hat{\mathfrak{s}}$
	In the holohomogeneous case, we have	
$L[y] \equiv M[y] - \lambda N[y] = 0$ (and homogeneous residual boundary conditions)		$\mathfrak{L}\,\mathfrak{a} \equiv \mathfrak{M}\,\mathfrak{a} - \Lambda\,\mathfrak{N}\,\mathfrak{a} = 0,$
	and thus	

(cont'd)

$$0 \leqq \lambda_i \leqq \Lambda_i \text{ for } i = 1, 2, 3, \dots, \varrho,$$

if

the differential expressions $M[y]$ and $N[y]$	the matrices $\mathfrak{M}$ and $\mathfrak{N}$

are symmetric and positive definite.

It should be specifically taken into consideration that the dynamic /152 inhomogeneity of the boundary conditions (individual forces, etc.) not only enters the energy term Π_d and thus the vector $\mathfrak{s}$ but also all matrices $\mathfrak{G}_\nu$ if comparison functions are selected (but not in the case of only admissible functions), whereas the inhomogeneity of the differential equation (tensile load, etc.) is expressed uniquely in the vector $\mathfrak{r}$. Conversely, a geometric inhomogeneity may appear in the energy $\tilde{\Pi}$ (for example, elastic support with prestressing) or may not appear in it (for example, permanent support sag); in any case, the inhomogeneity will enter the last rows of all matrices $\mathfrak{G}_\nu$ over the constant d [eq.(1.11)], no matter whether only admissible or comparison functions are selected.

2. Hermite Interpolation Polynomials

A Hermitian polynomial $\overset{2\alpha}{H}_i(\xi)$ of the order 2α (and thus of the degree 2α - - 1), is here defined by the property that, of the 2α values

$$\begin{cases} H_i(0),\ H_i'(0),\ H_i''(0), \dots, H_i^{(\alpha-1)}(0), \\ H_i(1),\ H_i'(1),\ H_i''(1), \dots, H_i^{(\alpha-1)}(1) \end{cases} \tag{2.1}$$

all vanish except for the i^{th} value which is equal to 1.

The totality of all Hermite polynomials of a permanently selected order 2α is written as follows:

$$\overset{2\alpha}{\mathfrak{p}}(\xi) = \overset{2\alpha}{\mathfrak{H}}\,\mathfrak{x} \tag{2.2}$$

with the coefficient matrix

and the vector

$$\overset{2\alpha}{\mathfrak{K}} = (\overset{2\alpha}{\mathfrak{t}}_0, \overset{2\alpha}{\mathfrak{t}}_1, \ldots, \overset{2\alpha}{\mathfrak{t}}_{2\alpha-1}) \tag{2.3}$$

$$\mathfrak{x} = (1, \xi, \xi^2, \ldots, \xi^{2\alpha-1}) \tag{2.4}$$

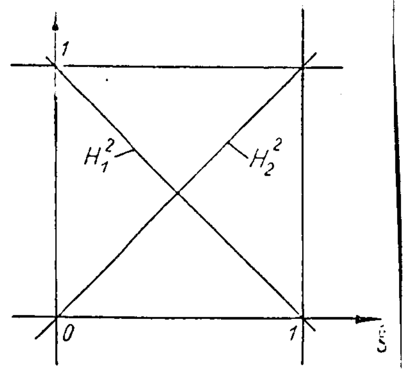

Fig.1 Hermite Polynomials of the Order $2\alpha = 2$ (Straight Lines)

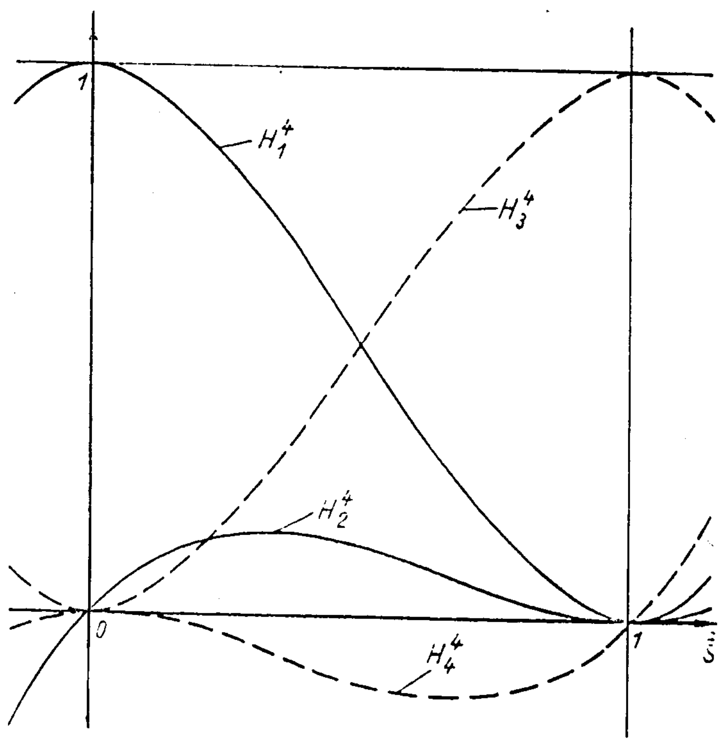

Fig.2 Hermite Polynomials of the Order $2\alpha = 4$ (Cubic Parabolas)

For $\alpha = 1, 2, 3$, and 4, the matrices $\overset{2\alpha}{\mathfrak{K}}$ are compiled in Table I. The course of all these polynomials in the - zero-free! - interval $0 \le \xi \le 1$ is plotted in Figs.1 - 4.

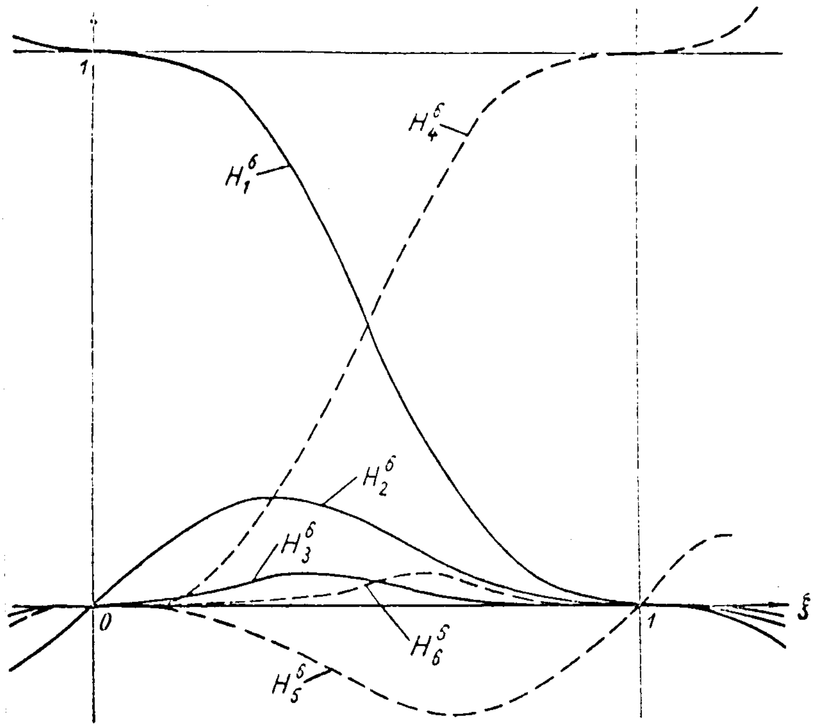

Fig.3 Hermite Polynomials of the Order $2\alpha = 6$
(Parabolas of the 5^{th} Degree)

It is known that the polynomial

$$\begin{cases} y(\xi) = y_0 H_1(\xi) + y_0' H_2(\xi) + \cdots + y_0^{(\alpha-1)} H_\alpha(\xi) \\ \qquad + y_1 H_{\alpha+1}(\xi) + y_1' H_{\alpha+2}(\xi) + \cdots + y_1^{(\alpha-1)} H_{2\alpha}(\xi) \end{cases} \tag{2.5}$$

has the property, exactly required for deriving an admissible or comparison function, of explicitly containing all boundary derivatives occurring in eq.(1.3). However, also the higher derivatives are readily obtained by a comparison with a Taylor series at the points $\xi = 0$ and $\xi = 1$. For the left-hand boundary, the following is valid: /153

$$y_0^{(n)} = n!\, \overset{2\alpha}{\mathfrak{f}}{}_n^* \overset{2\alpha}{\mathfrak{z}} \text{ for } n = 0, 1, 2, \ldots, 2\alpha - 1 \tag{2.6}$$

with

$$\overset{2\alpha}{\mathfrak{z}}{}^* = (y_0 \;\; y_0' \;\; y_0'' \ldots y_0^{(\alpha-1)};\; y_1 \;\; y_1' \;\; y_1'' \ldots y_1^{(\alpha-1)}) \tag{2.7}$$

and, similarly, for the left-hand boundary,

$$y_1^{(n)} = n!\, \overset{2\alpha}{\mathfrak{f}}{}_n^* \overset{2\alpha}{\mathfrak{w}} \text{ for } n = 0, 1, 2, \ldots . 2\alpha - 1 \tag{2.8}$$

with

$$\overset{2\alpha}{\mathfrak{w}}{}^* = (-1)^n (y_1 \;\; -y_1' \;\; y_1'' \ldots (-1)^{\alpha-1} y_1^{(\alpha-1)};\; y_0 \;\; -y_0' \;\; y_0'' \ldots (-1)^{\alpha-1} y_0^{(\alpha-1)}) . \tag{2.9}$$

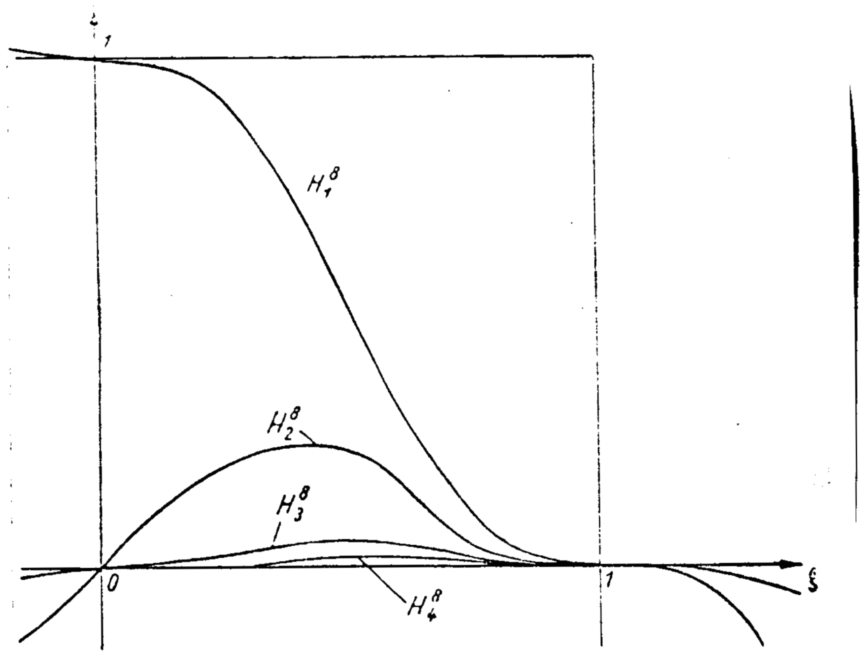

Fig.4 Hermite Polynomials of the Order $2\alpha = 8$
(Parabolas of the 7th Degree)

Consequently, here the indices 0 and 1 must be permuted with respect to /154 eq.(2.7), and the odd (even) derivatives for n = 0, 2, 4, 6...(n = 1, 3, 5, ...) must be provided with negative signs.

TABLE I

HERMITE POLYNOMIALS $\overset{2}{H}_i(\xi)$, $\overset{4}{H}_i(\xi)$, $\overset{6}{H}_i(\xi)$, and $\overset{8}{H}_i(\xi)$

$$\overset{2}{\mathfrak{p}}(\xi) = \begin{pmatrix} \overset{2}{H}_1 \\ \overset{2}{H}_2 \end{pmatrix} = \begin{array}{cc} \begin{array}{cc} 1 & \xi \end{array} & \\ \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} & \begin{array}{c} (y_0) \\ (y_1) \end{array} \end{array} ; \qquad \overset{4}{\mathfrak{p}}(\xi) = \begin{pmatrix} \overset{4}{H}_1 \\ \overset{4}{H}_2 \\ \overset{4}{H}_3 \\ \overset{4}{H}_4 \end{pmatrix} = \begin{array}{cccc|l} 1 & \xi & \xi^2 & \xi^3 & \\ \hline 1 & 0 & -3 & 2 & (y_0) \\ 0 & 1 & -2 & 1 & (y_0') \\ 0 & 0 & 3 & -2 & (y_1) \\ 0 & 0 & -1 & 1 & (y_1') \end{array}$$

$$\overset{6}{\mathfrak{p}}(\xi) = \begin{pmatrix} \overset{6}{H}_1 \\ \overset{6}{H}_2 \\ \overset{6}{H}_3 \\ \overset{6}{H}_4 \\ \overset{6}{H}_5 \\ \overset{6}{H}_6 \end{pmatrix} = \begin{array}{cccccc|l} 1 & \xi & \xi^2 & \xi^3 & \xi^4 & \xi^5 & \\ \hline 1 & 0 & 0 & -10 & 15 & -6 & (y_0) \\ 0 & 1 & 0 & -6 & 8 & -3 & (y_0') \\ 0 & 0 & \frac{1}{2} & -\frac{3}{2} & \frac{3}{2} & -\frac{1}{2} & (y_0'') \\ 0 & 0 & 0 & 10 & -15 & 6 & (y_1) \\ 0 & 0 & 0 & -4 & 7 & -3 & (y_1') \\ 0 & 0 & 0 & \frac{1}{2} & -\frac{2}{2} & \frac{1}{2} & (y_1'') \end{array}$$

$$\overset{8}{\mathfrak{p}}(\xi) = \begin{pmatrix} \overset{8}{H}_1 \\ \overset{8}{H}_2 \\ \overset{8}{H}_3 \\ \overset{8}{H}_4 \\ \overset{8}{H}_5 \\ \overset{8}{H}_6 \\ \overset{8}{H}_7 \\ \overset{8}{H}_8 \end{pmatrix} = \begin{array}{cccccccc|l} 1 & \xi & \xi^2 & \xi^3 & \xi^4 & \xi^5 & \xi^6 & \xi^7 & \\ \hline 1 & 0 & 0 & 0 & -35 & 84 & -70 & 20 & (y_0) \\ 0 & 1 & 0 & 0 & -20 & 45 & -36 & 10 & (y_0') \\ 0 & 0 & \frac{1}{2} & 0 & -\frac{10}{2} & \frac{20}{2} & -\frac{15}{2} & \frac{4}{2} & (y_0'') \\ 0 & 0 & 0 & \frac{1}{6} & -\frac{4}{6} & \frac{6}{6} & -\frac{4}{6} & \frac{1}{6} & (y_0''') \\ 0 & 0 & 0 & 0 & 35 & -84 & 70 & -20 & (y_1) \\ 0 & 0 & 0 & 0 & -15 & 39 & -34 & 10 & (y_1') \\ 0 & 0 & 0 & 0 & \frac{5}{2} & -\frac{14}{2} & \frac{13}{2} & -\frac{4}{2} & (y_1'') \\ 0 & 0 & 0 & 0 & -\frac{1}{6} & \frac{3}{6} & -\frac{3}{6} & \frac{1}{6} & (y_1''') \end{array}$$

3. Variation of the Boundary Values

We will construct our argument function (1.11) from nothing but Hermite polynomials of the same order 2α according to eq.(2.5) and then vary those 2n boundary values that do not vanish because of the boundary conditions (1.2) or that are linearly combined by the remaining conditions; the quantities a_i thus will have obtained a simple geometric or mechanical meaning. Despite the fact that the theory for the Ritz argument (1.11) merely stipulates admissible functions $v_i(x)$, numerically satisfactory results will be obtained only - specifically in multifield problems - if comparison functions are utilized: Such functions are those that satisfy not only the essential but all 2n boundary conditions (1.2).

Using the new dimensionless variables

$$\xi = \frac{x}{l}\,; \quad \frac{dy}{dx} = \frac{y'}{l}, \quad \frac{d^2y}{dx^2} = \frac{y''}{l^2}, \quad \frac{d^3y}{dx^3} = \frac{y'''}{l^3} \quad \text{etc.} \tag{3.1}$$

where the symbol ' denotes a derivative to ξ, we will have to establish for n = 1, i.e., for the differential equation

$$L[y] \equiv -(g_1(\xi)\, y'(\xi))' + g_0(\xi)\, y(\xi) = r(\xi) \tag{3.2}$$

the following argument

$$y(\xi) = y_0 \overset{4}{H}_1(\xi) + y_0' \overset{4}{H}_2(\xi) + y_1 \overset{4}{H}_3(\xi) + y_1' \overset{4}{H}_4(\xi)\,, \tag{3.3}$$

whereas for n = 2, i.e., for

$$L[y] \equiv (g_2(\xi)\, y''(\xi))'' - (g_1(\xi)\, y'(\xi))' + g_0\, y(\xi) = r(\xi) \tag{3.4}$$

the following argument must be used: /156

$$\left\{ \begin{aligned} y(\xi) = {} & y_0 \overset{8}{H}_1(\xi) + y_0' \overset{8}{H}_2(\xi) + y_0'' \overset{8}{H}_3(\xi) + y_0''' \overset{8}{H}_4(\xi) \\ & + y_1 \overset{8}{H}_5(\xi) + y_1' \overset{8}{H}_6(\xi) + y_1'' \overset{8}{H}_7(\xi) + y_1''' \overset{8}{H}_8(\xi) \end{aligned} \right. \tag{3.5}$$

Only in exceptional cases do the boundary conditions (1.2) require that, of the 4n quantities (1.3), simply half must be cancelled; in general, however, the boundary values must be linearly combined. This leads to the fact that, at each of the variables a_i, we have not only a Hermite polynomial but also a linear combination of the form

$$v_i(x) = b_1 H_1(\xi) + b_2 H_2(\xi) + \cdots + b_{4n} H_{4n}(\xi)\,, \tag{3.6}$$

or, in abbreviated form,

$$v_i(\xi) = \mathfrak{b}_i^* \, \mathfrak{p}(\xi) \tag{3.7}$$

Thus, the argument (1.12) will read

$$y(\xi) = \mathfrak{a}^* \, \mathfrak{w}(\xi) = \mathfrak{a}^* \, \mathfrak{B}^* \, \mathfrak{p}(\xi)\,, \tag{3.8}$$

TABLE II

INTEGRAL MATRICES FOR HERMITIAN POLYNOMIALS OF THE ORDERS $2\alpha = 2, 4, 6$

$$\int_0^1 \overset{2}{H}{}_i'' \, \overset{2}{H}{}_k'' \, d\xi \qquad \overset{2}{\mathfrak{H}}_{20} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad \int_0^1 \overset{4}{H}{}_i'' \, \overset{4}{H}{}_k'' \, d\xi \qquad \overset{4}{\mathfrak{H}}_{20} = \frac{840}{N_4} \begin{pmatrix} 6 & 3 & -6 & 3 \\ 3 & 2 & -3 & 1 \\ -6 & -3 & 6 & -3 \\ 3 & 1 & -3 & 2 \end{pmatrix}$$

$$\int_1^0 \overset{2}{H}{}_i' \, \overset{2}{H}{}_k' \, d\xi \qquad \overset{2}{\mathfrak{H}}_{10} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \qquad \int_0^1 \overset{4}{H}{}_i' \, \overset{4}{H}{}_k' \, d\xi \qquad \overset{4}{\mathfrak{H}}_{10} = \frac{14}{N_4} \begin{pmatrix} 36 & 3 & -36 & 3 \\ 3 & 4 & -3 & -1 \\ -36 & -3 & 36 & -3 \\ 3 & -1 & -3 & 4 \end{pmatrix}$$

$$\int_0^1 \overset{2}{H}_i \, \overset{2}{H}_k \, d\xi \qquad \overset{2}{\mathfrak{H}}_{00} = \frac{1}{6} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \qquad \int_0^1 \overset{4}{H}_i \, \overset{4}{H}_k \, d\xi \qquad \overset{4}{\mathfrak{H}}_{00} = \frac{1}{N_4} \begin{pmatrix} 156 & 22 & 54 & -13 \\ 22 & 4 & 13 & -3 \\ 54 & 13 & 156 & -22 \\ -13 & -3 & -22 & 4 \end{pmatrix}$$

$$\int_0^1 \overset{2}{H}_i \, d\xi \qquad \overset{2}{\mathfrak{h}}_0 = \frac{1}{2} \, (1 \quad 1) \qquad \int_0^1 \overset{4}{H}_i \, d\xi \qquad \overset{4}{\mathfrak{h}}_0 = \frac{35}{N_4} \, (6 \quad 1 \quad 6 \quad -1)$$

$$\int_0^1 \overset{6}{H}{}_i'' \, \overset{6}{H}{}_k'' \, d\xi \qquad \overset{6}{\mathfrak{H}}_{20} = \frac{792}{N_6} \cdot \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left(\begin{array}{rrrrrr} 1\,200 & 600 & 30 & -1\,200 & 600 & -30 \\ 600 & 384 & 22 & -600 & 216 & -8 \\ 30 & 22 & 6 & -30 & 8 & 1 \\ -1\,200 & -600 & -30 & 1\,200 & -600 & 30 \\ 600 & 216 & 8 & -600 & 384 & -22 \\ -30 & -8 & 1 & 30 & -22 & 6 \end{array}\right) \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}$$

$$\int_0^1 \overset{6}{H}{}_i' \, \overset{6}{H}{}_k' \, d\xi \qquad \overset{6}{\mathfrak{H}}_{10} = \frac{44}{N_6} \cdot \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left(\begin{array}{rrrrrr} 1\,800 & 270 & 15 & -1\,800 & 270 & -15 \\ 270 & 288 & 21 & -270 & -18 & 6 \\ 15 & 21 & 2 & -15 & -6 & 1 \\ -1\,800 & -270 & -15 & 1\,800 & -270 & 15 \\ 270 & -18 & -6 & -270 & 288 & -21 \\ -15 & 6 & 1 & 15 & -21 & 2 \end{array}\right) \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}$$

$$\int_0^1 \overset{6}{H}_i \, \overset{6}{H}_k \, d\xi \qquad \overset{6}{\mathfrak{H}}_{00} = \frac{1}{N_6} \cdot \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left(\begin{array}{rrrrrr} 21\,720 & 3\,732 & 281 & 6\,000 & -1\,812 & 181 \\ 3\,732 & 832 & 69 & 1\,812 & -532 & 52 \\ 281 & 69 & 6 & 181 & -52 & 5 \\ 6\,000 & 1\,812 & 181 & 21\,720 & -3\,732 & 281 \\ -1\,812 & -532 & -52 & -3\,732 & 832 & -69 \\ 181 & 52 & 5 & 281 & -69 & 6 \end{array}\right) \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}$$

$$\int_0^1 \overset{6}{H}_i \, d\xi \qquad \overset{6}{\mathfrak{h}}_0 = \frac{462}{N_6} \cdot (60 \quad 12 \quad 1 \quad 60 \quad -12 \quad 1)$$

$$N_6 = 5 \cdot 7 \cdot 9 \cdot 11 \cdot 16 = 55\,440$$

TABLE III

INTEGRAL MATRICES FOR HERMITIAN POLYNOMIALS OF THE ORDER $2\alpha = 8$

$$\int_0^1 \overset{8}{H}{}_i'' \, \overset{8}{H}{}_k'' \, d\xi \qquad \overset{8}{\mathfrak{H}}_{20} = \frac{936}{N_8} \quad \begin{array}{cc} \begin{array}{rrrrrrrr} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} & \\ \left(\begin{array}{rrrrrrrr} 352\,800 & 176\,400 & 16\,800 & 630 & -352\,800 & 176\,400 & -16\,800 & 630 \\ 176\,400 & 108\,000 & 11\,370 & 480 & -176\,400 & 68\,400 & -5\,430 & 150 \\ 16\,800 & 11\,370 & 3\,000 & 140 & -16\,800 & 5\,430 & -30 & -25 \\ 630 & 480 & 140 & 8 & -630 & 150 & 25 & -3 \\ -352\,800 & -176\,400 & -16\,800 & -630 & 352\,800 & -176\,400 & 16\,800 & -630 \\ 176\,400 & 68\,400 & 5\,430 & 150 & -176\,400 & 108\,000 & -11\,370 & 480 \\ -16\,800 & -5\,430 & -30 & 25 & 16\,800 & -11\,370 & 3\,000 & -140 \\ 630 & 150 & -25 & -3 & -630 & 480 & -140 & 8 \end{array}\right) & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} \end{array}$$

$$\int_0^1 \overset{8}{H}{}_i' \, \overset{8}{H}{}_k' \, d\xi \qquad \overset{8}{\mathfrak{H}}_{10} = \frac{18}{N_8} \quad \begin{array}{cc} \begin{array}{rrrrrrrr} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} & \\ \left(\begin{array}{rrrrrrrr} 1\,176\,000 & 227\,640 & 19\,320 & 700 & -1\,176\,000 & 227\,640 & -19\,320 & 700 \\ 227\,640 & 216\,000 & 22\,140 & 1000 & -227\,640 & 11\,640 & 2\,820 & -300 \\ 19\,320 & 22\,140 & 2\,920 & 148 & -19\,320 & -2\,820 & 980 & -73 \\ 700 & 1\,000 & 148 & 8 & -700 & -300 & 73 & -5 \\ -1176\,000 & -227\,640 & -19\,320 & -700 & 1\,176\,000 & -227\,640 & 19\,320 & -700 \\ 227\,640 & 11\,640 & -2\,820 & -300 & -227\,640 & 216\,000 & -22\,140 & 1\,000 \\ -19\,320 & 2\,820 & 980 & 73 & 19\,320 & -22\,140 & 2\,920 & -148 \\ 700 & -340 & -73 & -5 & -700 & 1\,000 & -148 & 8 \end{array}\right) & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} \end{array}$$

$$\int_0^1 \overset{8}{H}_i \, \overset{8}{H}_k \, d\xi \qquad \overset{8}{\mathfrak{H}}_{00} = \frac{1}{N_8} \quad \begin{array}{cc} \begin{array}{rrrrrrrr} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} & \\ \left(\begin{array}{rrrrrrrr} 5\,251\,680 & 978\,480 & 98\,640 & 4\,596 & 1\,234\,800 & -411\,480 & 55\,800 & -3\,126 \\ 978\,480 & 237\,600 & 26\,460 & 1\,296 & 411\,480 & -134\,280 & 17\,910 & -990 \\ 98\,640 & 26\,460 & 3\,096 & 156 & 55\,800 & -17\,910 & 2\,358 & -129 \\ 4\,596 & 1\,296 & 156 & 8 & 3\,126 & -990 & 129 & -7 \\ 1\,234\,800 & 411\,480 & 55\,800 & 3\,126 & 5\,251\,680 & -978\,480 & 98\,640 & -4\,596 \\ -411\,480 & -134\,280 & -17\,910 & -990 & -978\,480 & 237\,600 & -26\,460 & 1\,296 \\ 55\,800 & 17\,910 & 2\,358 & 129 & 98\,640 & -26\,460 & 3\,096 & -156 \\ -3\,126 & -990 & -129 & -7 & -4\,596 & 1\,296 & -156 & 8 \end{array}\right) & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} \end{array}$$

$$\int_0^1 \overset{8}{H}_i \, d\xi \qquad \overset{8}{\mathfrak{h}}{}_0^* = \frac{7722}{N_8}\,(840 \quad 180 \quad 20 \quad 1 \quad 840 \quad -180 \quad 20 \quad -1)$$

$$N_8 = 5 \cdot 7 \cdot 8 \cdot 9 \cdot 11 \cdot 13 \cdot 36 = 12\,972\,960; \quad 936 = 18 \cdot 52$$

where

$$\mathfrak{B} = (\mathfrak{b}_1, \mathfrak{b}_2, \ldots, \mathfrak{b}_{\varrho+1}) \tag{3.9}$$

represents a matrix with $\rho + 1$ columns and $2\alpha = 4n$ rows. Consequently, the integrals (1.16) and (1.17) have been transformed into

$$\mathfrak{G}_\nu = \int_0^1 g_\nu(\xi)\, \mathfrak{v}^{(\nu)}(\xi)\, \mathfrak{v}^{(\nu)*}(\xi)\, d\xi = \int_0^1 g_\nu(\xi)\, \mathfrak{B}^*\, \mathfrak{p}^{(\nu)}(\xi)\, \mathfrak{p}^{(\nu)*}(\xi) \cdot \mathfrak{B}\, d\xi = \mathfrak{B}^*\, \mathfrak{H}_\nu\, \mathfrak{B} \tag{3.10}$$

with

$$\mathfrak{H}_\nu = \int_0^1 g_\nu(\xi)\, \mathfrak{p}^{(\nu)}(\xi)\, \mathfrak{p}^{(\nu)*}(\xi) \cdot d\xi \tag{3.11}$$

and

$$\mathfrak{r} = \int_0^1 r(\xi)\, \mathfrak{v}(\xi)\, d\xi = \int r(\xi)\, \mathfrak{B}^* \mathfrak{p}(\xi)\, d\xi = \mathfrak{B}^* \mathfrak{h} \tag{3.12}$$

with

$$\mathfrak{h} = \int_0^1 r(\xi)\, \mathfrak{p}(\xi)\, d\xi\,. \tag{3.13}$$

Then, the computational scheme for the matrices $\mathfrak{G}_\nu$ and the vector $\mathfrak{r}$ will have the form

$$\begin{array}{ccc}
 & & \begin{array}{c}(a_1\, a_2\, \ldots\, a_0;\ d)\\ 2\alpha \left(\quad \mathfrak{B} \quad\right)\end{array} \\
2\alpha & \left(\quad \mathfrak{H}_\nu \quad\right)_{2\alpha} & \left(\quad \mathfrak{H}_\nu \mathfrak{B} \quad\right) \\
\varrho+1,\ \varrho & \left(\quad \mathfrak{B}^* \quad\right) & \underbrace{\left(\mathfrak{B}^* \mathfrak{H}_\nu \mathfrak{B} = \mathfrak{G}_\nu\right)}_{\varrho,\ \varrho+1}
\end{array}
\qquad
\boxed{\varrho = 2\alpha}
\qquad
\begin{array}{cc}
 & 2\alpha \left(\ \mathfrak{h}\ \right) \\
\left(\quad \mathfrak{B}^* \quad\right)_{2\alpha} & \left(\ \mathfrak{r}\ \right)\ \varrho,\ \varrho+1
\end{array}
\tag{3.14}$$

Since all matrices $\mathfrak{G}_\nu$ are symmetrical, only their right-hand upper portion need be calculated. Similarly, calculation of the last rows is not necessary since they are anyway eliminated in forming the gradient.

4. Computation of the Matrices $\mathfrak{H}_\nu$ and the Vector $\mathfrak{h}$

Calculation of the matrices $\mathfrak{H}_\nu$ with the elements

$$h_{\nu,ik} = h_{\nu,ki} = \int_0^1 g_\nu(\xi)\, H_i^{(\nu)}(\xi)\, H_k^{(\nu)}(\xi)\, d\xi \tag{4.1}$$

and of the vector $\mathfrak{h}$ with the components

$$h_i = \int_0^1 r(\xi)\, H_i(\xi)\, d\xi \tag{4.2}$$

is especially simple if the functions $g_\nu(\xi)$ and $r(\xi)$ are polynomials in ξ:

$$g_\nu(\xi) = g_{\nu 0} \cdot 1 + g_{1\nu} \cdot \xi + g_{\nu 2}\, \xi^2 \cdots + \mathfrak{G}_{\nu t}\, \xi^t\,, \tag{4.3}$$

$$r(\xi) = r_0 \cdot 1 + r_1 \xi + r_2 \xi^2 \cdots + r_j \xi^j . \tag{4.4}$$

This is so, since then

$$h_{\nu,ik} = g_{\nu 0} \int_0^1 H_i^{(\nu)} H_k^{(\nu)} \, d\xi + g_{\nu 1} \int_0^1 \xi \, H_i^{(\nu)} H_k^{(\nu)} \, d\xi + \cdots + g_{\nu t} \int_0^1 \xi^t H_i^{(\nu)} H_k^{(\nu)} \, d\xi , \tag{4.5}$$

$$h_i = r_0 \int_0^1 H_i \, d\xi + r_1 \int_0^1 \xi \, H_i \, d\xi + \cdots + r_j \int_0^1 \xi^j H_i \, d\xi , \tag{4.6}$$

and thus,

$$\mathfrak{H}_\nu = (h_{\nu,ik}) = g_{\nu 0} \mathfrak{H}_{\nu 0} + g_{\nu 1} \mathfrak{H}_{\nu 1} + \cdots + g_{\nu t} \mathfrak{H}_{\nu t} , \tag{4.7}$$

$$\mathfrak{h} = (h_i) = r_0 h_0 + r_1 \mathfrak{h}_1 + \cdots + r_j \mathfrak{h}_j . \tag{4.8}$$

The integral matrices $\overset{2\alpha}{\mathfrak{H}}_{20}$, $\overset{2\alpha}{\mathfrak{H}}_{10}$, $\overset{2\alpha}{\mathfrak{H}}_{00}$ and the vector $\overset{2\alpha}{\mathfrak{h}}_0$ are given in Tables II and III for α = 1, 2, 3, 4*. For example, according to Table II the element becomes

$$\overset{6}{h}_{10;35} = \int_0^1 \overset{6}{H'_3} \overset{6}{H'_5} \, d\xi = \frac{44}{N_6}(-6) = \frac{-44 \cdot 6}{5 \cdot 7 \cdot 9 \cdot 11 \cdot 16} = -\frac{1}{210} .$$

If the functions g_ν and r are no polynomials in ξ but are sufficiently ex- /158
actly - possibly, piecewise - replaceable by such (for example, again by Hermite polynomials), readymade integral matrices can be used also in this case; in the opposite case, the integration must be performed exactly or in first approximation.

5. The Eigenvalue λ Occurring in the Boundary Conditions

In holohomogeneous problems, the eigenvalue λ may occur linearly in the boundary condition and thus at most quadratically in the equivalent energy $\widetilde{\Pi} = \widetilde{\Pi}_d + \widetilde{\Pi}_k$. Consequently, the eigenvalue equation (1.26) assumes the form

$$L(\lambda)\,\mathfrak{a} \equiv [(\mathfrak{M}_0 + \lambda \mathfrak{M}_1 + \lambda^2 \mathfrak{M}_2) - \Lambda(\mathfrak{N}_0 + \lambda \mathfrak{N}_1 + \lambda^2 \mathfrak{N}_2)]\,\mathfrak{a} = 0 , \tag{5.1}$$

i.e., the coefficients s_μ of the characteristic polynomial (1.27) have now become rational functions of λ:

---

* Additional matrices will be published later (Bibl.2)

$$|\mathfrak{M}(\lambda) - \Lambda\,\mathfrak{N}(\lambda)| = s_\varrho(\lambda)\,\Lambda^\varrho + \cdots + s_2(\lambda)\,\Lambda^2 + s_1(\lambda)\,\Lambda + s_0(\lambda) = 0\,. \tag{5.2}$$

Since the argument functions $v_i(x)$ are admissible for any parameter value λ - for the true eigenvalues λ_i they even become exact comparison functions - eq.(1.28) is transformed into the more comprehensive statement:

$$\text{and any value of } \lambda \tag{5.3}$$

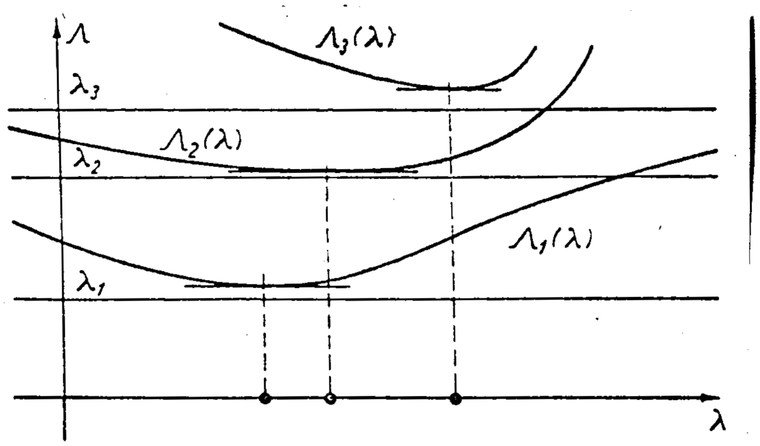

Fig.5 Characteristic Curves $\Lambda_i(\lambda)$

The ρ curve branches of the two-parametric eigenvalue problem (5.1) thus are limited downward by the wanted eigenvalues λ_i (see also Fig.5). Their extreme values relative to λ are obtained from the necessary condition

$$\frac{d\Lambda}{d\lambda} \equiv \dot{\Lambda} = 0\,. \tag{5.4}$$

Consequently, if we differentiate eq.(5.2) implicitly for λ and immediately put $\dot{\Lambda} = 0$, we obtain

$$\dot{s}_\varrho(\lambda)\,\Lambda^\varrho + \cdots + \dot{s}_2(\lambda)\,\Lambda^2 + \dot{s}_1(\lambda)\,\Lambda + \dot{s}_0(\lambda) = 0\,. \tag{5.5}$$

Next, the pair of equations (5.2) and (5.5) is multiplied successively by Λ, Λ^2, ..., $\Lambda^{\rho-1}$, yielding the homogeneous system of equations

$$
\mathfrak{S}\mathfrak{l} = \begin{array}{c} \begin{array}{cccccc} \Lambda^{2\varrho-1} & \Lambda^{2\varrho-2} & \Lambda^{2\varrho-3} & \Lambda^{2} & \Lambda^{1} & 1 \end{array} \\ \begin{pmatrix} s_\varrho & s_{\varrho-1} & s_{\varrho-2} \ldots\ldots & 0 & 0 & 0 \\ \dot{s}_\varrho & \dot{s}_{\varrho-1} & \dot{s}_{\varrho-2} \ldots\ldots & 0 & 0 & 0 \\ 0 & s_\varrho & s_{\varrho-1} \ldots\ldots & 0 & 0 & 0 \\ 0 & \dot{s}_\varrho & \dot{s}_{\varrho-1} \ldots\ldots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 \quad \ldots\ldots & s_2 & s_1 & s_0 \\ 0 & 0 & 0 \quad \ldots\ldots & \dot{s}_2 & \dot{s}_1 & \dot{s}_0 \end{pmatrix} \end{array} = 0. \qquad (5.6)
$$

Therefore, the vanishing of the determinant of $\mathfrak{S}$ is a necessary condition for the wanted parameter values λ. For practical computation, $\mathfrak{S}$ is first brought to the upper triangular form $\tilde{\mathfrak{S}}$ by rearrangement of the rows, after which the value λ_μ pertaining to Λ_μ is taken from the next to the last row of $\tilde{\mathfrak{S}}$.

For $\rho = 1$ and $\rho = 2$, we can give the solution directly in explicit form:

$$
\varrho = 1: \quad -\Lambda = \frac{s_0}{s_1} = \frac{\dot{s}_0}{\dot{s}_1}, \qquad (5.7)
$$

$$
\varrho = 2: \quad -\Lambda = \frac{s_0\,\dot{s}_2 - \dot{s}_0\,s_2}{s_1\,\dot{s}_2 - \dot{s}_1\,s_2} = \frac{s_0\,\dot{s}_1 - \dot{s}_0\,s_1}{s_0\,\dot{s}_2 - \dot{s}_0\,s_2}. \qquad (5.8)
$$

Here it must be considered that, in forming the difference, the highest powers of λ are eliminated in both numerator and denominator. If the eigenvalue λ occurs only a single time outside of the differential equation, i.e., if the argument (1.11) has the form of

$$
y(x) = a_1\,v_1(x) + \cdots + a_\varrho\,v_\varrho(x) + \lambda_i\,a_i\,v_i(x)\,, \qquad (5.9)
$$

we can put simply $\lambda\ a_1 = a_{\rho+1}$ and thus obtain a ρ + 1-row argument which no longer contains λ.

For $\rho > 2$, the described solution process is quite cumbersome; therefore, it is more convenient to calculate the corresponding root Λ from eq.(5.2) for several estimated values of λ in the vicinity of a suspected minimum and then to select the smallest of the resultant values as the optimum approximation value. Another method consists in putting $\lambda = \Lambda$; however, according to eq.(5.1), this will lead to equations of the degree 3ρ in Λ, without yielding especially good approximations.

6. Arguments with Less than 2n Variables a_i

Up to now, we assumed that a differential equation of the order $2n$ is coordinated with an argument containing Hermite polynomials of the order $2\alpha = 4n$ with $\rho = 2n = \alpha$ variables a_i. For example, the differential equation (3.4) thus always leads to a system of equations with $2n = 4$ unknowns whose solution, specifically in the holohomogeneous case, is somewhat tedious, a fact which is the more inconvenient as it is frequently desired to approximate only the smallest eigenvalue λ_1; however, this can already be obtained with an ordinary Rayleigh quotient. To decrease the number of variables, either the order 2α of the Hermite polynomials can be reduced or they are retained and arbitrary relations between the $2n$ variables a_i are created; expressed differently, a new Ritz argument is derived for the equivalent problem. Both methods will be described briefly below.

a) The Order 2α is Smaller than $4n$

Here, the highest derivatives $y_0^{(2n-1)}$ or $y_1^{(2n-1)}$ can no longer be directly covered by the argument (1.11). However, if these derivatives also occur in the boundary conditions (1.2) and if one does not want to restrict the computation to only admissible functions - in principle, this way out is not accepted by us in what follows - all necessary higher derivatives are found from eq.(2.6) or eq.(2.8) as linear combinations of the low derivatives so that, also in this case, the derivation of comparison functions offers no difficulty.

b) Ritz Argument for the Equivalent Problem

Here, we retain the order $2\alpha = 4n$ but combine the $\rho = 2n$ variables a_i, in a suitable manner, into $\sigma < \rho$ new variables c_i:

$$\mathfrak{a} = \mathfrak{C}\,\mathfrak{c} \tag{6.1}$$

with

$$\mathfrak{a}^* = (a_1, a_2, \ldots, a_\rho; d) \tag{6.2}$$

and

$$\mathfrak{c}^* = (c_1, c_2, \ldots, c_\sigma; d), \tag{6.3}$$

where the rectangular matrix $\mathfrak{C}$ has $\sigma + 1$ columns and $\rho + 1$ rows; for this reason, in the computational scheme (3.14) ρ must be replaced by σ and the matrix $\mathfrak{B}$ by the matrices $\mathfrak{B}\mathfrak{C}$.

In itself, one can arbitrarily dispose of the variables a_i - for example, equate all except one to zero - since any linear combination of comparison functions again constitutes a comparison function, which specifically means that the statement (1.28) remains valid for any selection of a_i. However, useful numerical results are obtained only if the information given by the posed problem (1.1), (1.2) is exhausted more than before: The simplest way is to do this by continuous differentiation of the differential equation (1.1) which then, together with the boundary conditions (1.2), will yield arbitrarily many boundary conditions with higher than 2n - 1-th derivatives; these, however, according to eqs.(2.6) and (2.8) are known linear combinations of the boundary values (1.3), which means that the wanted relations between the 2n variables a_i are found. This procedure even permits to select the order 2α of the Hermite polynomials higher than 4n (as in the example No.2), so that now even a single-term argument permits raising the accuracy to as high a value as desired. Incidentally, this constitutes a method known and proved valuable for long in special cases as "insertion of an iteration".

Another type of linear combination becomes possible if the solution of a differential equation is available in the exact or approximate form and if the same problem is to be solved again with slightly modified values, as is frequently the case in technical problems. Then, the 2α boundary derivatives of the known solution are used as "framework" for the modified problem, yielding rather satisfactory results in general by using only a single-term argument (Rayleigh quotient). Such frameworks or matrices can be used advantageously also if a problem with Hermite polynomials of a certain order - or else with arbitrary different argument functions - had been calculated and if the order is to be increased subsequently. This yields not only better results but the calculation also is freed of all rounding-off and computational errors made in the first passage. /160

Finally, it should be mentioned that, in symmetric problems (as in the example No.1), the calculation can always be subdivided into a symmetrical and an antimetric component by a suitable combination of the variables a_i.

7. Arguments with More than 2n Variables a_i

If, in the holohomogeneous case, more than the first 2n eigenvalues λ_i are to be approximated, eq.(1.27) must be of the degree $\rho > 2n$. This is achieved simplest by subdividing the field of the length ℓ into 2, 3, ..., f regions and by requiring a steady transition of $y, y', y'' \dots y^{2n-1}$ at the interfaces. In this manner, eq.(1.26) becomes a system of equations of the order 4n, 6n, ... 2fn which now permits the calculation of correspondingly many approximation values $\Lambda_i \geqq \lambda_i$. However, since multifield problems will be discussed only in a later report (Bibl.2), discussion of this method will be postponed until then.

8. Examples

Example 1*. Given is the differential equation

$$g_2 y'''' - g_1 y'' + g_0 y = 0 \tag{8.1}$$

with constant coefficients g_2, g_1, g_0 and the boundary conditions

$$y_0 = y_0'' = y_1 = y_1'' = 0 . \tag{8.2}$$

The corresponding energy expressions are

$$2\,\Pi_k = g_2 \int_0^1 y''^2\, d\xi + g_1 \int_0^1 y'^2\, d\xi + g_0 \int_0^1 y^2\, d\xi ; \quad \Pi_a = 0 . \tag{8.3}$$

Thus, the equivalent energy, in accordance with eq.(1.20), will be

$$2\,\tilde{\Pi} = \mathfrak{a}^* (g_2\, \mathfrak{G}_{20} + g_1\, \mathfrak{G}_{10} + g_0\, \mathfrak{G}_{00})\, \mathfrak{a} , \tag{8.4}$$

from which, as a finite transformation of eq.(8.1), it follows that

$$\operatorname{grad} \tilde{\Pi} = (g_2\, \mathfrak{G}_{20} + g_1\, \mathfrak{G}_{10} + g_0\, \mathfrak{G}_{00})\, \mathfrak{a} = 0 . \tag{8.5}$$

First Approximation

We will put an argument with Hermite polynomials of the sixth order according to eq.(2.5):

$$y(\xi) = y_0 \overset{6}{H}_1(\xi) + y_0' \overset{6}{H}_2(\xi) + y_0'' \overset{6}{H}_3(\xi) + y_1 \overset{6}{H}_4(\xi) + y_1' \overset{6}{H}_5(\xi) + y_1'' \overset{6}{H}_6(\xi) , \tag{8.6}$$

of which, because of eq.(8.2), only the two polynomials H_2 and H_5 remain:

$$y(\xi) = y_0' \overset{6}{H}_2(\xi) + y_1' \overset{6}{H}_5(\xi) \tag{8.7}$$

which means that the elements with the index pairs 22, 25, 52, and 55 must be singled out from the matrices $\overset{6}{\mathfrak{H}}_{\nu o}$; according to Table II, this will yield

$$\left\{ \frac{792}{N_6} \cdot g_2 \begin{pmatrix} 384 & 216 \\ 216 & 384 \end{pmatrix} + \frac{44}{N_6}\, g_1 \begin{pmatrix} 288 & -18 \\ -18 & 288 \end{pmatrix} + \frac{1}{N_6}\, g_0 \begin{pmatrix} 832 & -532 \\ -532 & 832 \end{pmatrix} \right\} \mathfrak{a} = 0 ,$$

or, in combined form,

---

* All examples are taken from the thesis by G.Brüne (Bibl.3).

$$\frac{4}{N_6}\cdot\begin{array}{c}\begin{array}{cc} \qquad\qquad y_0' \qquad\qquad & \qquad\qquad y_1' \qquad\qquad \end{array}\\ \begin{pmatrix} 76032\,g_2 + 3168\,g_1 + 208\,g_0 & 42768\,g_2 - 198\,g_1 - 133\,g_0 \\ 42768\,g_2 - 198\,g_1 - 133\,g_0 & 76032\,g_2 + 3168\,g_1 + 208\,g_0 \end{pmatrix}\end{array} = \begin{pmatrix}0\\0\end{pmatrix}. \tag{8.8}$$

The determinant of this matrix has the form /161

$$\Delta\cdot\left(\frac{N_6}{4}\right)^2 = \begin{vmatrix} a & b \\ b & a \end{vmatrix} = a^2 - b^2 = (a+b)(a-b) = 0 \tag{8.9}$$

and, consequently, resolves into the two factors

$$a + b = 118800\,g_2 + 2970\,g_1 + 75\,g_0 = 0\,, \tag{8.10}$$

$$a - b = 33264\,g_2 + 3366\,g_1 + 341\,g_0 = 0\,. \tag{8.11}$$

Second Approximation

Next, we put the argument with Hermite polynomials of the eighth order:

$$y(\xi) = y_0'\,\overset{8}{H}_2(\xi) + y_0'''\,\overset{8}{H}_4(\xi) + y_1'\,\overset{8}{H}_6(\xi) + y_1'''\,\overset{8}{H}_8(\xi)\,, \tag{8.12}$$

where we have already canceled y_0, y_0'', y_1, y_1'' according to eq.(8.2). Therefore, the following four quantities will now be varied:

$$a_1 = y_0'\,,\quad a_2 = y_0'''\,,\quad a_3 = y_1'\,,\quad a_4 = y_1'''\,. \tag{8.13}$$

Thus, in the matrices $\overset{8}{\mathfrak{H}}_{20}$, $\overset{8}{\mathfrak{H}}_{10}$, and $\overset{8}{\mathfrak{H}}_{00}$, only the first, third, fifth, and seventh rows and columns must be eliminated, thus directly yielding the equivalent problem (8.5); we then have

$$\mathfrak{G}_{20} = \frac{936}{N_8}\cdot\begin{pmatrix} 108\,000 & 480 & 68\,400 & 150 \\ 480 & 8 & 150 & -3 \\ 68\,400 & 150 & 108\,000 & 480 \\ 150 & -3 & 480 & 8 \end{pmatrix}, \tag{8.14}$$

$$\mathfrak{G}_{10} = \frac{18}{N_8}\cdot\begin{pmatrix} 216\,000 & 1\,000 & 11\,640 & -300 \\ 1\,000 & 8 & -300 & -5 \\ 11\,640 & -300 & 216\,000 & 1\,000 \\ -300 & -5 & 1\,000 & 8 \end{pmatrix}, \tag{8.15}$$

$$\mathfrak{G}_{00} = \frac{1}{N_8}\cdot\begin{pmatrix} 237\,600 & 1\,296 & -134\,280 & -990 \\ 1\,296 & 8 & -990 & -7 \\ 134\,280 & -990 & 237\,600 & 1\,296 \\ -990 & -7 & 1\,296 & 8 \end{pmatrix}. \tag{8.16}$$

Because of the holosymmetry of the boundary conditions (8.2), the problem can be resolved into a symmetric portion with

$$y_0' = -y_1' \qquad \text{and} \qquad y_0''' = -y_1''' \tag{8.17}$$

and into an antimetric portion with

$$y_0' = y_1' \qquad \text{and} \qquad y_0''' = y_1''' , \tag{8.18}$$

(see also Fig.6). In this manner, two multirow matrix triples are obtained which, each separately, approximate the first and third or the second and fourth

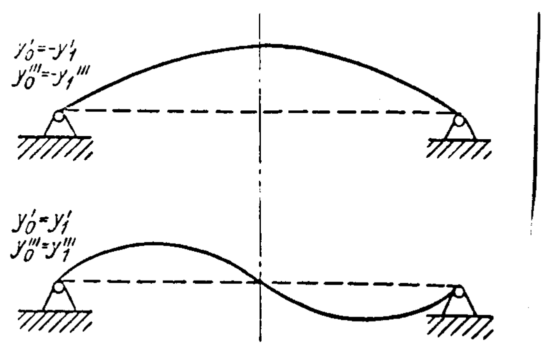

Fig.6 Symmetric and Antisymmetric Eigenfunctions for Example No.1

eigenvalue; results are given in Table IV.

Third Approximation

Again, we use the argument (8.12) but make also use of the fact that, in addition to eq.(8.2), the following is valid because of eq.(8.1):

$$y_0'''' = y_1'''' = 0 \tag{8.19}$$

in order to express the two variables y_0''' and y_0''' by y_0' and y_1'. First, eq.(2.6) according to Table I yields

$$y_0'''' = 41 \, \overset{8}{\mathfrak{f}}{}_4^* \, \overset{\circ}{\delta} , \tag{8.20}$$

where $\overset{8}{\mathfrak{k}}_4$ represents the fourth column of the coefficient matrix $\overset{8}{\mathfrak{K}}$, namely,

$$\overset{8}{\mathfrak{k}}_4^* = \left(-35 \quad -20 \quad -5 \quad -\frac{2}{3} \quad 35 \quad -15 \quad \frac{5}{2} \quad -\frac{1}{6}\right) \tag{8.21}$$

Using the vector $\overset{8}{\mathfrak{z}}^*$ (2.7)

$$\overset{8}{\mathfrak{z}}^* = (y_0 \quad y_0' \quad y_0'' \quad y_0''' \quad y_1 \quad y_1' \quad y_1'' \quad y_1''') \tag{8.22}$$

eq.(8.20), because of eq.(8.2) will read, in a more extensive form, /162

$$y_0'''' = 4!\left(-20\, y_0' - \frac{2}{3}\, y_0''' - 15\, y_1' - \frac{1}{6}\, y_1'''\right) = 0\,. \tag{8.23}$$

Next, eq.(2.8) is utilized

$$y_1'''' = 4!\, \overset{8}{\mathfrak{k}}_4^*\, \overset{8}{\mathfrak{w}} \quad \text{with} \quad \overset{8}{\mathfrak{w}}^* = (y_1 \quad -y_1' \quad y_1'' \quad -y_1''';\ y_0 \quad -y_0' \quad y_0'' \quad -y_0''') \tag{8.24}$$

in accordance with eq.(2.9). Together with eq.(8.21) and based on eqs.(8.2) and (8.19), this will yield

$$y_1'''' = 4!\left(20\, y_1' + \frac{2}{3}\, y_1''' + 15\, y_0' + \frac{1}{6}\, y_0'''\right) = 0\,, \tag{8.25}$$

It is then easy to calculate from eqs.(8.23) and (8.25):

$$y_0''' = -26\, y_0' - 16\, y_1'\,; \quad y_1''' = -16\, y_0' - 26\, y_1'\,, \tag{8.26}$$

which, substituted in eq.(8.12), yields the argument function

$$y(\xi) = y_0'\left(\overset{8}{H}_2 - 26\, \overset{8}{H}_4 - 16\, \overset{8}{H}_8\right) + y_1'\left(-16\, \overset{8}{H}_4 + \overset{8}{H}_6 - 26\, \overset{8}{H}_8\right) \tag{8.27}$$

with the two variables $a_1 = y_0'$, $a_2 = y_1'$ and thus the matrix (3.9)

$$\mathfrak{B}^* = \begin{pmatrix} \mathfrak{b}_1^* \\ \mathfrak{b}_2^* \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & -26 & 0 & 0 & 0 & -16 \\ 0 & 0 & 0 & -16 & 0 & 1 & 0 & -26 \end{pmatrix} \tag{8.28}$$

by means of which the final transformation matrices

$$\left\{\begin{aligned} \mathfrak{G}_{20} &= \mathfrak{B}^*\, \mathfrak{H}_{20}\, \mathfrak{B} = \begin{pmatrix} 83\,200 & 49\,100 \\ 49\,100 & 83\,200 \end{pmatrix} \cdot \frac{936}{N_8}\,, \\ \mathfrak{G}_{10} &= \mathfrak{B}^*\, \mathfrak{H}_{10}\, \mathfrak{B} = \begin{pmatrix} 176\,896 & -2\,764 \\ -2\,764 & 176\,896 \end{pmatrix} \cdot \frac{18}{N_8}\,, \end{aligned}\right. \tag{8.29}$$

$$\mathfrak{G}_{00} = \mathfrak{B}^* \mathfrak{H}_{00} \mathfrak{B} = \begin{pmatrix} 203\,520 & -124\,140 \\ -124\,140 & 203\,520 \end{pmatrix} \cdot \frac{1}{N_8}$$

can be calculated. The determinant

$$\Delta = |g_2 \mathfrak{G}_{20} + g_1 \mathfrak{G}_{10} + g_0 \mathfrak{G}_{00}| = 0 \tag{8.30}$$

because of the double symmetry, is resolved as in eq.(8.9) into the two equations

$$\left.\begin{aligned} 31\,917\,600\, g_2 + 3\,233\,880\, g_1 + 327\,660\, g_0 = 0\,, \\ 123\,832\,800\, g_2 + 3\,134\,376\, g_1 + 79\,380\, g_0 = 0\,. \end{aligned}\right\} \tag{8.31}$$

For a better comparison, the approximations (8.10), (8.11), (8.31), and the first two solutions obtained from eqs.(8.14) - (8.16), were divided by the factor at g_0; see Table IV. The exact solution reads

$$y(\xi) = \sin n\pi\xi \quad \text{with } g_2(n\pi)^4 + g_1(n\pi)^2 + g_0 = 0 \quad \text{for} \quad n = 1, 2, 3, \ldots, \infty. \tag{8.32}$$

Aside from this, eqs.(8.1) and (8.2) contain numerous technically important problems such as the elastically supported beam harmonically vibrating under constant pressure (two-parametric eigenvalue problem).

TABLE IV

APPROXIMATION VALUES AND EXACT SOLUTION FOR EXAMPLE NO.1

$g_2 y'''' - g_1 y'' + g_0 y = 0$ $y_0 = y_0'' = y_1 = y_0'' = 0$	$n = 1$			$n = 2$		
	g_2	g_1	g_0	g_2	g_1	g_0
$\overset{6}{H_i}$ with $a_1 = y_0', a_2 = y_1'$	97.548 387	9.870 968	1	1584.0000	39.600 000	1
$\overset{8}{H_i}$ with $a_1 = y_0', a_2 = y_1'$	97.410 731	9.869 621	1	1560.0000	39.485 714	1
$\overset{8}{H_i}$ with $a_1 = y_0', a_2 = y_0''', a_3 = y_1', a_3 = y_1'''$	97.409 137	9.869 046	1	1558.6394	39.478 656	1
Exact Solution	97.409 091	9.869 044	1	1558.5455	39.478 418	1

Example 2. Given is the differential equation /163

$$ \tag{8.33} $$

with the boundary conditions

$$y_0' = 0\,, \quad y_1 = 0\,. \tag{8.34}$$

The corresponding variational problem reads

$$\Pi_k = \frac{1}{2}\int_0^1 (y'^2 + y^2 - 2\,y)\,d\xi \Rightarrow \text{Extremum}\,, \tag{8.35}$$

and the exact solution is

$$y(\xi) = 1 - \frac{\cosh\xi}{\cosh 1} = 0{,}351\,945\,727 - 0{,}324\,027\,137\,\xi^2 - 0{,}027\,002\,262\,\xi^4 - 0{,}000\,900\,075\,\xi^6 - 0{,}000\,016\,073\,\xi^8 - \cdots\,. \tag{8.36}$$

Despite the fact that an argument with the cubic polynomials $\overset{4}{H}_1(\xi)$ would be sufficient for including the boundary conditions (8.34), we will perform a two-term argument with Hermite polynomials of the order $2\alpha = 8$. In this case, of the eight boundary derivatives in eq.(3.5), because of eq.(8.34), only two will vanish first; in order to eliminate four additional derivatives of the remaining six, any four relations between these will be required; we select

$$y_0''' = 0\,; \qquad y_1'' = -1 \tag{8.37}$$

and

$$y_0^{(5)} = 0\,; \qquad y_1^{(4)} = -1\,, \tag{8.38}$$

which are equations that are readily obtained by differentiation of eq.(8.33), i.e., from

$$-y''' + y' = 0\,, \qquad -y'''' + y'' = 0 \tag{8.39}$$

together with eq.(8.34). Consequently, making use of eqs.(8.34) and (8.37), all that remains of the argument (3.5) is

$$y(\xi) = y_0\,\overset{8}{H}_1(\xi) + y_0''\,\overset{8}{H}_3(\xi) + y_1'\,\overset{8}{H}_6(\xi) + (-1)\,\overset{8}{H}_7(\xi) + y_1'''\,\overset{8}{H}_8(\xi) \tag{8.40}$$

and the boundary values y_0'' and y_1''', occurring here, can be eliminated then by means of eq.(8.38) in accordance with eqs.(2.6) and (2.8):

$$y_0^{(5)} = 5!\,\overset{8}{\mathfrak{k}}{}_5^*\,\overset{8}{\mathfrak{z}} = 5!\cdot\left(84\,y_0 + 45\,y_0' + 10\,y_0'' + 1\cdot y_0''' - 84\,y_1 + 39\,y_1' - 7\,y_1'' + \frac{1}{2}\,y_1'''\right) \tag{8.41}$$

$$= 5!\cdot\left(84\,y_0 + 10\,y_0 + 39\,y_1 + 7 + \frac{1}{2}\,y_1\right) = 0\,,$$

$$y_1^{(4)} = 4!\,\overset{8}{\mathfrak{k}}{}_4^*\,\overset{8}{\mathfrak{w}} = 4!\cdot\left(-35\,y_1 - 20\,(-y_1') - 5\,y_1'' - \frac{2}{3}\,(-y_1''') + 35\,y_0 - 15\,(-y_0')\right. \tag{8.42}$$

$$\left. + \frac{5}{2}\,y_0'' - \frac{1}{6}\,(-y_0''')\right)$$

$$= 4!\cdot\left(20\,y_1' + 5 + \frac{2}{3}\,y_1''' + 35\,y_0 + \frac{5}{2}\,y_0''\right) = -1\,,$$

where we already have used the two equations (8.34) and (8.37). From eqs.(8.41) and (8.42), after a brief calculation, we obtain

$$260\,y_0'' = -1\,848\,y_0 - 768\,y_1' - 103\,, \tag{8.43}$$

$$260\,y_1''' = -6\,720\,g_0 - 4\,920\,y_1' - 1\,580\,, \tag{8.44}$$

Substituting this in eq.(8.40) will furnish the final argument function

$$\left\{\begin{aligned} &260\,y_{(\xi)} = 4\,y_0\,(65\,\overset{8}{H}_1 - 462\,\overset{8}{H}_3 - 1680\,\overset{8}{H}_8) + 4\,y_1'\,(-192\,\overset{8}{H}_3 + 65\,\overset{8}{H}_6 - 1230\,\overset{8}{H}_8) \\ &\qquad\qquad + (-103\,\overset{8}{H}_3 - 260\,\overset{8}{H}_7 - 1580\,\overset{8}{H}_8) \\ &\text{or} \\ &\qquad\qquad\qquad y(\xi) = a_1\,v_1(\xi) + a_2\,v_2(\xi) + 1\cdot v_3(\xi) \end{aligned}\right. \tag{8.45}$$

in accordance with eq.(3.6), with

$$a_1 = 4\,y_0\,,\quad a_2 = 4\,y_1'\,,\quad d = 1\,, \tag{8.46}$$

[see also eq.(1.11)]. Thus, the matrix $\mathfrak{B}$ (3.9) becomes /164

$$\mathfrak{B}^* = \frac{1}{260}\begin{pmatrix} 65 & 0 & -462 & 0 & 0 & 0 & 0 & -1\,680 \\ 0 & 0 & -192 & 0 & 0 & 65 & 0 & -1\,230 \\ 0 & 0 & -103 & 0 & 0 & 0 & -260 & -1\,580 \end{pmatrix}, \tag{8.47}$$

after which, according to the scheme (3.14) and Table III, the matrices

$$\mathfrak{G}_1 = \mathfrak{B}^* \overset{8}{\mathfrak{H}}_{10} \mathfrak{B} = \frac{18}{260^2 N_8} \begin{pmatrix} 4\,187\,877\,120 & 850\,702\,020 & 272\,609\,120 \\ 850\,702\,020 & 908\,353\,920 & 333\,836\,570 \\ 272\,609\,120 & 333\,836\,570 & * \end{pmatrix} \tag{8.48}$$

$$\mathfrak{G}_0 = \mathfrak{B}^* \overset{8}{\mathfrak{H}}_{00} \mathfrak{B} = \frac{1}{260^2 N_8} \begin{pmatrix} 17\,429\,900\,544 & -2\,147\,049\,396 & -1\,015\,197\,984 \\ -2\,147\,049\,396 & 1\,308\,968\,064 & 532\,102\,056 \\ -1\,015\,197\,984 & 532\,102\,056 & * \end{pmatrix} \tag{8.49}$$

and the vector

$$\mathfrak{r} = \mathfrak{B}^* \overset{8}{\mathfrak{h}}_0 = \frac{1}{260\, N_8} \begin{pmatrix} 363\,242\,880 \\ -110\,501\,820 \\ * \end{pmatrix} \tag{8.50}$$

are calculated, in which case the second, fourth, fifth, and seventh rows (or components) of the matrices $\mathfrak{H}_{10}\mathfrak{B}$ and $\mathfrak{H}_{00}\mathfrak{B}$ (or of the vector $\mathfrak{h}$) are not even needed. In addition, for symmetry reasons only five elements need be determined in eqs.(8.48) and (8.49) and only two elements in eq.(8.50); the elements denoted by an asterisk are of no interest at all since they vanish together with the last rows in forming the gradient.

Then, the finite transformation of eq.(8.35) to eq.(1.15) will read

$$\widetilde{\Pi}_k = \frac{1}{2}\, \mathfrak{a}^* (\mathfrak{G}_1 + \mathfrak{G}_0)\, \mathfrak{a} - \mathfrak{a}^* \mathfrak{r} \Rightarrow \text{Extremum}\,, \tag{8.51}$$

i.e.,

$$\operatorname{grad} \widetilde{\Pi}_k = \frac{2}{2} (\mathfrak{G}_1 + \mathfrak{G}_0)\, \mathfrak{a} - \mathfrak{r} = 0\,. \tag{8.52}$$

Expressed in numerals, we have

$$\frac{1}{260^2 N_8} \begin{array}{c} \begin{array}{ccc} a_1 & a_2 & 1 \end{array} \\ \begin{pmatrix} 92\,811\,688\,704 & 13\,165\,586\,964 & -90\,551\,382\,624 \\ 13\,165\,586\,964 & 17\,659\,338\,624 & 35\,262\,633\,516 \end{pmatrix} \end{array} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{8.53}$$

The solutions

$$a_1 = 4\, y_0 = 1.407\,782\,351\,; \qquad a_2 = 4\, y_1' = -3.046\,371\,93 \tag{8.54}$$

are substituted in eq.(8.45), yielding

$$\begin{aligned} y(\xi) = {} & 0.351\,945\,587 - 0.324\,023\,144\, \xi^2 - 0.027\,022\,666\, \xi^4 \\ & - 0.000\,842\,410\, \xi^6 - 0.000\,057\,367\, \xi^7\,. \end{aligned} \tag{8.55}$$

A comparison with eq.(8.36) shows the exceptional quality of this approximation; the fifth power of ξ is exactly canceled out and the seventh power, in the mean, represents the remainder of the terminated infinite series. In Table V, several values of $y(\xi)$, $y'(\xi)$, and $y''(\xi)$ are compiled. The function itself is reproduced correctly to within six places, the first derivative to within five

TABLE V

SEVERAL FUNCTIONAL VALUES FOR EXAMPLE NO.2

		exact	Approximation
$y(\xi)$	0.0	0.351 945 73	0.351 945 59
	0.2	0.338 941 38	0.338 941 37
	0.4	0.299 406 43	0.299 406 56
	0.6	0.231 754 20	0.231 754 21
	0.8	0.133 269 57	0.133 269 43
	1.0	0	0
$y'(\xi)$	0.0	0	0
	0.2	—0.130 476 66	—0.130 475 77
	0.4	—0.266 189 80	—0.266 189 72
	0.6	—0.412 586 08	—0.412 587 13
	0.8	—0.575 540 88	—0.575 540 97
	1.0	—0.761 594 15	—0.761 592 98
$y''(\xi)$	0.0	—0.648 054 28	—0.648 046 29
	0.2	—0.661 058 62	—0.661 058 37
	0.4	—0.700 593 57	—0.700 601 39
	0.6	—0.768 245 80	—0.768 246 85
	0.8	—0.866 730 43	—0.866 721 41
	1.0	—1.0	—1.0

places, and the second derivative to within four to five places.

Example 3. The following differential equation refers to the vibration system of Fig.7:

$$\eta'''' + \lambda^4 \eta = 0 ; \qquad \lambda^4 = \frac{\mu \omega^2 l^4}{E J} \tag{8.56}$$

with the boundary conditions

$$\eta_0 = \eta_0' = \eta_1'' = 0 \tag{8.57}$$

and

$$\eta_1''' = (3 - 1 \cdot \lambda^4)\, \eta_1 , \tag{8.58}$$

where

$$x = l\,\xi , \quad w = l\,\eta ; \quad \frac{dw}{dx} = \eta' , \quad \frac{d^2w}{dx^2} = l\,\eta'' , \quad \frac{d^3w}{dx^3} = l^2 \eta''' , \quad \frac{d^4w}{dx^4} = l^3 \eta'''' \tag{8.59}$$

had been assumed. The corresponding energies are

$$\frac{2\,l}{E\,J}\,\Pi_k = \frac{l}{E\,J}\left(\int_0^l E\,J\left(\frac{d^2 w}{dx^2}\right)^2 dx - \omega^2 \int_0^l \mu\, w^2\, dx\right) = \int_0^1 \eta''^2\, d\xi - \lambda^4 \int_0^1 \eta^2\, d\xi\,, \tag{8.60}$$

$$\frac{2\,l}{E\,J}\,\Pi_a = \frac{l}{E\,J}\,(c\,w_1^2 - m\,\omega^2\,w_1^2) = (3 - 1\cdot\lambda^4)\,\eta_1^2\,. \tag{8.61}$$

We select Hermite polynomials of the sixth order, of which three are immediately eliminated because of eq.(8.57). This will leave

$$\eta(\xi) = \eta_0''\,\overset{6}{H}_3(\xi) + \eta_1\,\overset{6}{H}_4(\xi) + \eta_1'\,\overset{6}{H}_5(\xi)\,. \tag{8.62}$$

Next, we satisfy eq.(8.58) and the auxiliary condition $\eta_0''' = 0$ which, because of $\eta_0 = 0$, follows from the differential equation. According to eqs.(2.6)

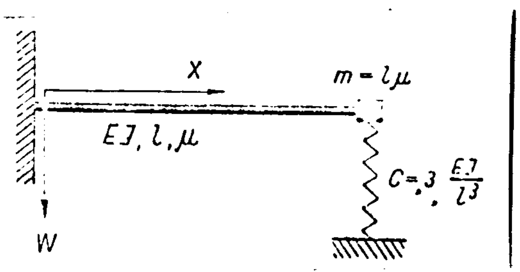

Fig.7 Clamped Beam with End Load and Spring;
Example No.3

and (2.8) and Table I, this results in

$$l^3\left(\frac{d^3 w}{dx^3}\right)_1 = \eta_0''' = 3!\,\overset{6}{\mathfrak{k}}{}_3^*\,\overset{6}{\mathfrak{w}} = 3!(-1)^3\left(-10\,\eta_1 - 6\,(-\eta_1') - \frac{3}{2}\,\eta_1'' + 10\,\eta_0 - 4\,(-\eta_0') + \frac{1}{2}\eta_0''\right) \tag{8.63}$$

$$= -24\left(-10\,\eta_1 + 6\,\eta_1' + \frac{1}{2}\,\eta_0''\right) = (3 - \lambda^4)\,\eta_1\,.$$

$$l^3\left(\frac{d^4 w}{dx^4}\right)_0 = \eta_0'''' = 4!\,\overset{6}{\mathfrak{k}}{}_4^*\,\overset{6}{\mathfrak{d}} = 4!\left(15\,\eta_0 + 8\,\eta_0' + \frac{3}{2}\,\eta_0'' - 15\,\eta_1 + 7\,\eta_1' - 1\,\eta_1''\right) \tag{8.64}$$

$$= 4!\left(\frac{3}{2}\,\eta'' - 15\,\eta + 7\,\eta'\right) = 0\,,$$

from which it follows that

$$66\,\eta_0'' = [240 - 14\,(3 - \lambda^4)]\,\eta_1\,; \qquad 66\,\eta_1' = [90 + 3\,(3 - \lambda^4)]\,\eta_1\,. \tag{8.65}$$

Substituting this into eq.(8.62) will yield the final argument function

$$\eta(\xi) = \left(6\,\overset{6}{H}_3 + 2\,\overset{6}{H}_4 + 3\,\overset{6}{H}_5\right)\cdot\frac{1}{2}\,\eta_1 + \left(14\,\overset{6}{H}_3 - 3\,\overset{6}{H}_5\right)\frac{\lambda^4}{66}\cdot\eta_1 \tag{8.66}$$

with

$$= \mathfrak{b}_1^* \overset{6}{\mathfrak{p}}(\xi)\, a_1 + \mathfrak{b}_2^* \overset{6}{\mathfrak{p}}(\xi)\, a_2 \tag{8.67}$$

and

$$\mathfrak{B}^* = \begin{pmatrix} \mathfrak{b}_1^* \\ \mathfrak{b}_2^* \end{pmatrix} = \begin{pmatrix} 0 & 0 & 6 & 2 & 3 & 0 \\ 0 & 0 & 14 & 0 & -3 & 0 \end{pmatrix} \tag{8.68}$$

$$a_1 = \eta_1/2\,, \qquad a_2 = \eta_1 \cdot \lambda^4/66\,; \qquad \mathfrak{a}^* = (a_1\, a_2)\,, \tag{8.69}$$

in which manner, according to eq.(5.9), the eigenvalue λ^4 is eliminated from the argument. Thus, according to eq.(8.61) and (8.60), the equivalent energies will be /166

$$\frac{2\,l}{E\,J}\,\widetilde{\Pi}_d = (3 - \Lambda^4)\,\eta_1^2 = (12 - 4\,\Lambda^4)\,a_1^2 = \mathfrak{a}^*\,\mathfrak{F}\,\mathfrak{a} \tag{8.70}$$

with

$$\mathfrak{F} = \begin{pmatrix} 12 - 4\,\Lambda^4 & 0 \\ 0 & 0 \end{pmatrix} \tag{8.71}$$

and

$$\frac{2\,l}{E\,J}\,\widetilde{\Pi}_k = \mathfrak{a}^*\,(\mathfrak{B}^*\,\mathfrak{H}_{20}\,\mathfrak{B} - \Lambda^4\,\mathfrak{B}^*\,\mathfrak{H}_{00}\,\mathfrak{B})\,\mathfrak{a} = \mathfrak{a}^*\,\mathfrak{G}\,\mathfrak{a} \tag{8.72}$$

with

$$\mathfrak{G} = \frac{792}{N_6}\begin{pmatrix} 840 & 0 \\ 0 & 3969 \end{pmatrix} - \frac{\Lambda^4}{N_6}\begin{pmatrix} 52\,272 & 19\,228 \\ 19\,228 & 13\,032 \end{pmatrix} = \frac{12}{7}\begin{pmatrix} 7 & 0 \\ 0 & 33 \end{pmatrix} - \frac{\Lambda^4}{13\,860}\begin{pmatrix} 13\,068 & 4\,807 \\ 4\,807 & 3\,258 \end{pmatrix}. \tag{8.73}$$

In this case, because of the many zeros in $\mathfrak{B}$ (8.68), the calculation of the two matrices $\mathfrak{B}^*\mathfrak{H}_{20}\mathfrak{B}$ or $\mathfrak{B}^*\mathfrak{H}_{00}\mathfrak{B}$ according to Table II requires only 23 multiplications each (instead of 90 in the full matrix $\mathfrak{B}$). If then we introduce the new parameter

$$\varepsilon = \frac{7}{12} \cdot \frac{\Lambda^4}{13\,860} = \frac{\Lambda^4}{23\,760} \tag{8.74}$$

the finite transformation of the differential equation (8.56) will read

$$\frac{7}{12}\,(\mathfrak{F} - \mathfrak{G})\,\mathfrak{a} = (\mathfrak{M} - \varepsilon\,\mathfrak{N})\,\mathfrak{a} = 0 \tag{8.75}$$

while the determinant

$$|\mathfrak{M} - \varepsilon\,\mathfrak{N}| = \begin{vmatrix} 14 - 68\,508\,\varepsilon & -4\,807\,\varepsilon \\ -4\,807\,\varepsilon & 33 - 3\,258\,\varepsilon \end{vmatrix} = 0 \tag{8.76}$$

equated to zero will yield the quadratic equation

$$200\,091\,815\,\varepsilon^2 - 2\,306\,376\,\varepsilon + 462 = 0 \tag{8.77}$$

with the roots ϵ_1, ϵ_2 from which, according to eq.(8.74), it follows that

$$\begin{cases} \Lambda^4 = 4{,}845\,185\,16\,; & \Lambda_1 = 1{,}483\,637 > \lambda_1 = 1{,}483\,633\,; \ \textbf{error} \ 0{,}0003\%\,, \\ \Lambda^4 = 269{,}026\,555\,; & \Lambda_2 = 4.049\,942 > \lambda_2 = 4{,}032\,159\,; \ \textbf{error} \ 0{,}4\%\,. \end{cases} \tag{8.78}$$

The exact values are solutions of the frequency equation

$$\begin{cases} E(\lambda) + \dfrac{\beta - \alpha\lambda^4}{\lambda^3} B(\lambda) = 0\,; & \alpha = \dfrac{m}{\mu l} = 1\,, \quad \beta = \dfrac{c\,l^3}{EJ} = 3\,; \\ E(\lambda) = \cosh\lambda \cos\lambda + 1\,; & B(\lambda) = \cosh\lambda \sin\lambda - \sinh\lambda \cos\lambda\,. \end{cases} \tag{8.79}$$

BIBLIOGRAPHY

1. Collatz, L.: Numerical Treatment of Differential Equations (Numerische Behandlung von Differentialgleichungen). Springer, pp.132 - 151, 1951.
2. - The Rayleigh-Ritz Method with Hermitian Interpolation Polynomials in Multifield Problems (Das Verfahren von Rayleigh-Ritz mit Hermiteschen Interpolationspolynomen bei mehrfeldrigen Aufgaben). ZAMM 1964.
3. Brüne, G.: The Ritz Method in Technical Eigenvalue Problems with Hermitian Polynomials (Ritz-Verfahren bei technischen Eigenwertaufgaben mit Hermiteschen Polynomen). Polytechnic Institute Braunschweig, 1962.

Manuscript submitted: 16 February, 1962

Address: Prof.Dr.S.Falk, Department for Mechanics and Stress Analysis of the Polytechnic Institute Braunschweig.